- **5338:** *Proposed by Arkady Alt, San Jose, CA*

Determine the maximum value of

$$F(x,y,z)=\min\left\{\frac{|y-z|}{|x|},\frac{|z-x|}{|y|},\frac{|x-y|}{|z|}\right\},$$

where x,y,z are arbitrary nonzero real numbers.

Solution 1 by Kee-Wai Lau, Hong Kong, China

We show that the maximum value of $F(x,y,z)$ is 1.

We first prove that

$$F(x,y,z)\le 1, \qquad (1)$$

by showing that at least one of the numbers $\dfrac{|y-z|}{|x|},\dfrac{|z-x|}{|y|},\dfrac{|x-y|}{|z|}$ is less than equal to 1.

Suppose, on the contrary, that all of them are greater than 1. From $\dfrac{|y-z|}{|x|}>1$, we obtain

$$(y-z)^2>x^2,\ \text{or}\ (x+y-z)(x-y+z)<0. \qquad (2)$$

Similarly from $\dfrac{|z-x|}{|y|}>1$, and $\dfrac{|x-y|}{|x|}>1$, we obtain respectively

$$(x-y-z)(x+y-z)>0, \qquad (3)$$

and

$$(x-y-z)(x-y+z)>0. \qquad (4)$$

Multiplying $(2),(3)$ and (4) together. we obtain

$$(x+y-z)^2(x-y+z)^2(x-y-z)^2<0,$$

which is false. Thus (1) holds. Since $F(2,-1,1)=1$, we see that the maximum value of $F(x,y,z)$ is 1 indeed.

Solution 2 by Albert Stadler, Herrliberg, Switzerland

We claim that the maximum value equals 1.

Let $x>0$. Then $F(x,x+1,-1)=\min\left\{\dfrac{x+2}{x},\dfrac{x+1}{x+1},\dfrac{1}{1}\right\}=1.$

So the maximum value is ≥ 1.

Suppose the maximum value is >1. Then there is a triple (x,y,z) with

$$|y-z|>|x|,\ |z-x|>|y|,\ |x-y|>|z|. \qquad (1)$$

By cyclic symmetry, we can assume that $x\le\min(y,z)$.

Assume first that $x\le y\le z$. Then (1) reads as

$z-y>|x|,\ z-x>|y|,\ y-x>|z|$. So $z-x=(z-y)+(y-x)>|x|+|z|\ge z-x$

which is a contradiction.

Assume next that $x \le z \le y$. Then (1) reads as

$y - z > |x|, \; z - x > |y|, \; y - x > |z|$. So $y - x = (y - z) + (z - x) > |x| + |y| \ge y - x$, which is a contradiction.

This concludes the proof.

Solution 3 by Paolo Perfetti, Department of Mathematics, "Tor Vergata" Unversity, Rome Italy

Answer: 1

The symmetry of $F(x, y, z)$ allows us to set $z \le y \le x$. We have two cases:

$$\begin{aligned} &1) \;\; 0 < z \le y \le x \text{ and} \\ &2) \;\; z < 0, \; 0 < y \le x. \end{aligned}$$

Moreover, by observing that $F(x, y, z) = F(-x, -y, -z)$, the case $z \le y < 0$, $x > 0$ is recovered by the case 2) simply changing sign to all the signs and the same happens if $z \le y \le x < 0$.

Now we study the case 1)

$$\frac{|y-z|}{|x|} \le \frac{|x-z|}{|y|} \iff \frac{y-z}{x} \le \frac{x-z}{y} \iff z \le x + y$$

which evidently holds true. Moreover,

$$\frac{|y-z|}{|x|} \le \frac{|x-y|}{|z|} \iff \frac{y-z}{x} \le \frac{x-y}{z} \iff yx + yz \le x^2 + z^2$$

This generates two subcases.

1.1) $0 < z \le y \le x$ and $yx + yz \le x^2 + z^2$. In this case we must find the maximum of the function $\dfrac{y-z}{x}$. We have

$$\frac{y-z}{x} \le \frac{y-z}{y} = 1 - \frac{z}{y} < 1.$$

The value 1 is not attained because $z \ne 0$.

1.2) $0 < z \le y \le x$ and $yx + yz > x^2 + z^2$. In this case we must find the maximum of the function $\dfrac{x-y}{z}$. We have

$$\frac{x-y}{z} < \frac{y-z}{x} \le \frac{y-z}{y} = 1 - \frac{z}{y} < 1.$$

Now we study case 2)

$$F(x, y, z) = \min\left\{\frac{y-z}{x}, \frac{x-z}{y}, \frac{x-y}{-z}\right\}$$

and

$$\frac{y-z}{x} \leq \frac{x-z}{y} \iff z \leq x+y$$

which evidently holds true.
Moreover,

$$\frac{y-z}{x} \leq \frac{z-y}{-z} \iff y \leq x+z.$$

This generates two subcases.

2.1) $z<0$, $0<y<x$, $y \leq x+z$. In this case we must find the maximum of

$$\frac{y-z}{x} \leq \frac{x}{x} = 1.$$

The maximum achieved.

2.2) $z<0$, $0<y<x$, $y>x+z$. In this case we must find the maximum of

$$\frac{x-y}{-z} \leq \frac{x-y}{x-y} = 1.$$

The maximum achieved.

Also solved by Jerry Chu, (student at Saint George's School), Spokane, WA; Ethan Gegner, (student, Taylor University), Upland, IN, and the proposer.

- **5339:** *Proposed by D.M. Bătinetu-Giurgiu, "Matei Basarab" National College, Bucharest, Romania and Neculai Stanciu "George Emil Palade" School, Buzău, Romania*

Calculate: $\displaystyle\int_0^{\pi/2} \frac{3\sin x + 4\cos x}{3\cos x + 4\sin x} dx.$

Solution 1 by Haroun Meghaichi (student, University of Science and Technology Houari Boumediene), Algeria

Consider the general case for $a, b > 0$:

$$I(a,b) = \int_0^{\pi/2} \frac{a\sin x + b\cos x}{b\sin x + a\cos x}\,\mathrm{d}x,$$

Note that the derivative of the denominator (with respect to x) is $b\cos x - a\sin x$, and $\{b\sin x + a\cos x, b\cos x - a\sin x\}$ form a base on $R[\cos x, \sin x]$, then there are $\alpha, \beta \in R$ such that

$$a\sin x + b\cos x = \alpha\left(b\sin x + a\cos x\right) + \beta\left(b\cos x - a\sin x\right), \quad \forall\, x \in R$$

$$\Leftrightarrow \quad b - a\alpha - b\beta = a - b\alpha + a\beta = 0.$$

We can easily solve the system to get $(\alpha, \beta) = \left(\dfrac{2ab}{a^2+b^2}, \dfrac{b^2-a^2}{a^2+b^2}\right)$, then

$$I(a,b) = \frac{1}{a^2+b^2}\int_0^{\pi/2} 2ab + (b^2-a^2)\frac{b\cos x - a\sin x}{b\sin x + a\cos x}\,\mathrm{d}x$$